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# The optimality of the velocity-gradient method in the problem of controlling the escape from a potential well☆

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**Abstract**

The problem of controlling the escape of a particle from a potential well for a nonlinear system with friction is considered. The velocity-gradient method [Polushin IG, Fradkov AL, Hill, D. Passivity and passivation in non-linear systems. *Avtomatika i Telemekhanika* 2000;**3**:3–37] is proved to be optimal in the sense that if it does not guarantee escape from the well, then this is also impossible with any other control law. Nonlinear Duffing and Helmholtz oscillators with one degree of freedom and negative stiffness are considered. For each of them a curve is constructed separating the parameter plane of the problem into two parts: one where escape is feasible and one where it is not. An estimate is obtained for the inclination angle of the tangent to that curve near the origin.


## 1. Introduction

Along with the problem of suppression of oscillations in a system[2], there is a large class of problems in which, as a rule, one needs to maintain oscillations at a given energy level, or is concerned with the excitation of an oscillatory system from an initial state to a required one, or else with escape from potential wells. Such problems are common in the design of vibrational machines and mechanisms in which the working parts perform reciprocating or reciprocating rotational motion[3,4], generators of electrical or acoustic oscillations, etc.

The problem of the excitability of a system to a given level is usually formulated as that of transferring the system from an initial stable equilibrium position to a new state. In particular, in engineering one frequently encounters the problem of transferring a system from one equilibrium position to another. Here we consider the escape of a system from a potential well (or overcoming a potential barrier), which may serve as an intermediate stage in considering the transfer from one equilibrium position to another and also as an independent treatment arising in many technical areas (the excitation of membranes and shells, the capsizing of ships, etc)[5,6]. Often, passage through a potential barrier corresponds to a phase transition in a physical system.

Unlike the well-known Duffing oscillator with a term that is cubic in the coordinate for describing the stiffness of a spring, which is encountered in many mechanical problems, below we consider an analogous system with negative

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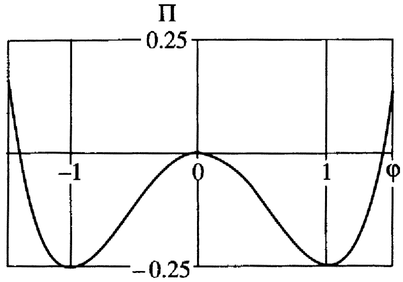


Fig. 1.

linear stiffness. The dynamic equation takes the form

$$\ddot{\varphi} + \rho\dot{\varphi} + \Pi'(\varphi) = u, \quad |u| \le \gamma \tag{1.1}$$

where the potential is the Duffing potential

$$\Pi(\varphi) = -\varphi^2/2 + \varphi^4/4$$

and the control parameter is subject to a symmetrical constraint (Fig. 1). The dissipation and the resistance in the surrounding medium are taken into account by a linear term containing the coefficient of friction ρ. The quantity γ, characterizing a bound of the perturbation, and the coefficient ρ are regarded to be small enough for the oscillatory character of the motion to be preserved.

This system is the simplest possible model for the forced oscillations of a cantilever beam in an inhomogeneous field of two permanent magnets. Fig. 2 shows the scheme for an experiment[7]: a thin steel beam $B$ is fixed in a rigid body with magnets $M_l$ and $M_r$ fastened to it. It has been shown[7] that Eq. (1.1) describes the dynamics of a beam or plate when one considers only one mode of oscillations, which is justified for a sufficiently thin and long beam, and also for powerful magnets (then the observed oscillations correspond primarily to the first mode). The control in that case is provided by the exciting force.

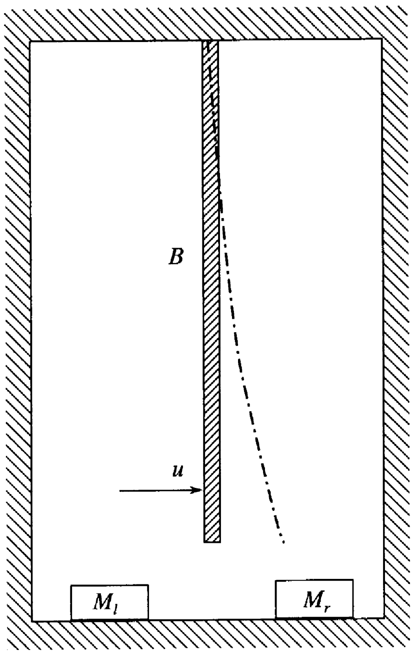


Fig. 2.

Various control algorithms have been compared[8] in relation to transferring system (1.1) from one equilibrium position to another. We will consider the problem of determining the range of $\gamma$ and $\rho$ values in which that transition can be performed, or, in other words, it is possible to escape from the potential well. A numerical study has shown that the gradient control method $u = \gamma\,\mathrm{sign}\dot{\varphi}$ produces better results than the harmonic law of action $u = \gamma\sin\omega t$. In particular, when $\rho = 0.25$, escape is inevitable if $\gamma > 0.212$, $\omega \approx 1.07$, whereas it does not occur for any value of the external driving frequency for $\gamma < 0.212$. On the other hand, excitation with feedback leads to escape at smaller amplitudes. For example, for $\rho = 0.25$, an estimate of the critical value gives $\gamma = 0.1767$. Numerical simulation has shown that escape from a potential well is also possible at much smaller exciting amplitudes, $\gamma \approx 0.1225$.

The following problems arise: how far does the gradient control method (control using the velocity gradient) agree with the method that is globally optimal; namely, is escape possible with smaller amplitudes when one uses another control method? Can one obtain a more accurate estimate of the parameters for which escape occurs? This paper deals with the answers to these questions.

System (1.1) has been examined in many papers. For instance, it was considered in Refs. 9 and 10 from the viewpoint of the theory of dynamical systems and the theory of bifurcations under conditions of harmonic excitation or a constant excitation law, without feedback control. A more detailed study has been performed[8] using the velocity-gradient method (VG method), which claimed local optimality[11], while the problem of global optimality was considered as unresolved. Here we show that the VG method is in fact globally optimal, i.e., if it does not guarantee attainment of the control purpose, then that is impossible with any other control algorithm. Proof of this is given by means of an additional consideration of optimal control with a free right-hand end. At the end of this paper, we refine the boundaries of the region in parameter space in which escape from a potential well is feasible.

Oscillatory system control has also been examined in other papers such as Ref. 12; there is a discussion in Ref. 13 of the excitation of a linear oscillator to a required energy level, which can be applied to escape from a potential well. However, global optimality of the VG method has been proved[13] by explicit solution of the auxiliary optimal control problem by means of Pontryagin's maximum principle[14]. This is not possible in the general case of a nonlinear system such as a Duffing oscillator or a mathematical pendulum.

## 2. The velocity-gradient method[1]

The problem is solved by the velocity-control method[8], the essence of which is as follows. Consider a non-stationary control system

$$\dot{x} = F(x, u, t), \quad u \in U$$

Let the aim of the control $u$ be the attainment of the equality

$$\lim_{t \to \infty} Q(x(t), t) = 0$$

where $Q(x, t)$ is a smooth objective function. Then, to construct the control algorithm, we calculate the scalar function $Q' = \omega(x, u, t)$, the rate of change of $\bar{Q}(t) = Q(x(t), t)$ by virtue of the equation of motion

$$\omega(x, u, t) = \partial Q(x, t)/\partial t + [\nabla_x Q(x, t)]^T F(x, u, t)$$

Then we derive the gradient of the function $\omega(x, u, t)$ with respect to the input variable, the control $u$:

$$\nabla_u \omega(x, u, t) = [\partial\omega/\partial u]^{\mathrm{T}} = [\partial F/\partial u]^{\mathrm{T}} \nabla_x Q(x, t)$$

which is defined by the algorithm for the change in $u(t)$ or according to the differential equation

$$du/dt = -\Gamma\nabla_u \omega(x, u, t)$$

where $\Gamma = \Gamma^T > 0$ is a symmetrical positive-definite matrix, or in accordance with the final relation

$$u(t) = -\gamma\,\mathrm{sign}\nabla_u \omega(x, u, t)$$

in the case of bang-bang control constraints. This has been called the velocity gradient algorithm, since the change in the control $u(t)$ is proportional to the rate of change in $\bar{Q}(t)$.

As an example we consider the excitation of a pendulum

$$\ddot{\varphi} + \sin\varphi = u, \quad |u| \le \gamma$$

to a given energy level $H_*$ in which the objective function may conveniently be taken as

$$Q(x) = (H_0(\varphi, \psi) - H_*)^2/2$$

Here $H_0(\varphi, \psi)$ is the Hamiltonian for the unperturbed case ($u=0$), $\psi = \dot{\varphi}$. Then the relations for the VG method may be written in the simpler form

$$u = -\gamma\,\mathrm{sign}((H_0 - H_*)\dot{\varphi})$$

If it is known in advance that $H_0 < H_*$, then that law amounts to bang-bang velocity control:

$$u = \gamma\,\mathrm{sign}\,\dot{\varphi} \tag{2.1}$$

## 3. Investigation of the basic problem

We will examine system (1.1) with a potential function of general form $\Pi : \mathbb{R} \to \mathbb{R}$. At the initial instant, the system is in a state of stable equilibrium $(\varphi_0, 0)$, determined by the condition for the minimum of the potential: $\Pi'(\varphi_0) = 0$, $\Pi''(\varphi_0) > 0$. Each equilibrium position can be put into correspondence with its basin of attraction, which is usually constructed for the case of nondissipative uncontrolled systems ($\rho = 0$, $u = 0$); this basin of attraction is bounded by parts of a separatrix. We say that escape from the potential well is possible if a control $u(t)$ exists such that the corresponding path $\varphi(t)$ intersects the separatrix at a certain instant and subsequently lies outside the basin of attraction of the point $(\varphi_0, 0)$. It is assumed that the basin of attraction does not contain other equilibrium positions or limit cycles.

For conservative systems, escape can occur for any non-zero boundary controls. This is not always possible when there is dissipation; for example, for large $\rho$ and for small $\gamma$, escape is impossible. Consequently, for each value of $\rho \neq 0$ a critical value $\gamma^*$ exists, so that escape is possible for $\gamma > \gamma^*$, while for $\gamma < \gamma^*$ the point will remain indefinitely within the basin of attraction, no matter what control algorithm is used.

The problem of escape from a potential well is essentially a game of quality with one player. According to the standard method[15], its solution requires a formulation of an additional problem. As such an additional problem, we consider here an optimal control problem with a free right end, namely, the problem of reducing system (1.1) at the boundary (terminal) of the set $\Omega = \{(\varphi, \psi)\colon \varphi - \varphi_1 = 0\}$ having a certain terminal functional, whose specific form is not important. Here $\varphi_1$ is the point of intersection of the branches of the separatrix. It is possible to bring the phase point to the boundary of the region $\Omega$ if and only if escape is possible.

The Hamiltonian for the auxiliary problem is written in the form

$$\mathcal{H}(\varphi, \psi, p, q, u) = p\psi + q(-\Pi'(\varphi) - \rho\psi + u)$$

in which $\psi = \dot{\varphi}$ and $p$ and $q$ are a pair of conjugate variables. The system of equations for the maximum principle is as follows:

$$\dot{\varphi} = \psi, \quad \dot{\psi} = -\Pi'(\varphi) - \rho\psi + u, \quad \dot{p} = q\Pi''(\varphi), \quad \dot{q} = -p + \rho q \tag{3.1}$$

Pontryagin's maximum principle implies that the optimal control is defined from the condition for a maximum of the Hamiltonian:

$$u = \underset{|u| \le \gamma}{\arg\max}\,\mathcal{H}(\varphi, \psi, p, q, u) = \gamma\,\mathrm{sign}\,q \tag{3.2}$$

It remains to specify the initial conditions for (3.1); they are written as $\varphi(0) = \varphi_0$, $\psi(0) = 0$, while the remaining two conditions are defined from the conditions for transversality at the boundary of the set $\Omega$.

The main result is formulated as the following theorem.

**Theorem 1.** *Suppose we are given a system of general form (1.1) with initial position at the point of stable equilibrium* $(\varphi_0, 0)$: $\Pi'(\varphi_0) = 0$, $\Pi''(\varphi_0) > 0$. *Suppose the basin of attraction of the point* $(\varphi_0, 0)$ *does not contain other equilibrium positions or limit cycles. Then for sufficiently small* $\rho$ *and* $\gamma$, *it is possible to have controlled escape from the potential well if and only if it is possible using the bang-bang control method of the form (2.1).*

**Proof.** We will assume that escape from the well is possible, i.e., the auxiliary problem has a solution. We will show that algorithm (3.2) is identical with the control method (2.1), i.e., $\mathrm{sign}\,q=\mathrm{sign}\,\psi$. As the system is autonomous, the condition applies when the Hamiltonian of the problem vanishes identically along the optimal path:

$$\mathcal{H}(\varphi(t), \psi(t), p(t), q(t), u(t)) \equiv 0$$

or, omitting the argument $t$ in this expression and using (3.1), it can be rewritten as

$$\mathcal{H}(\varphi, \psi, p, q) = p\dot{\varphi} + q\dot{\psi} = (\rho q - \dot{q})\psi + q\dot{\psi} = 0 \tag{3.3}$$

Consider time intervals when the control is constant and takes one of the extreme values $+\gamma$ or $-\gamma$. Then all the time functions in the expression are smooth. We divide Eq. (3.3) by the product $q\psi$ and we obtain

$$\frac{\dot{q}}{q} - \frac{\dot{\psi}}{\psi} = \rho \Rightarrow \frac{d}{dt}\ln\frac{q}{\psi} = \rho$$

Hence it follows that $q(t) = C\psi(t)e^{\rho t}$, where $C$ takes a definite fixed value only in the interval between instants of switching the control $u$. It follows from this expression that the zeros of the two functions $q(t)$ and $\psi(t)$ coincide.

At the switching instant, the derivative of $\psi(t)$ has a discontinuity, but $q(t)$ and its derivative remain continuous throughout the interval of motion of the system. Then, writing an expression for the derivative of $q$

$$\dot{q} = \rho q + C\dot{\psi}e^{\rho t}$$

we obtain that at the switching instant $t=\tau$ the following equality must be satisfied

$$C^+\dot{\psi}^+ - C^-\dot{\psi}^- = 0; \quad \dot{\psi}^{\pm} = -\Pi'(\varphi(\tau)) \pm \gamma$$

where $C^+$ and $C^-$ are the values of $C$ at adjacent time intervals with the control $u=\gamma$ and $u=-\gamma$. Then

$$C^+ = C^-[\Pi'(\varphi(\tau)) + \gamma]/[\Pi'(\varphi(\tau)) - \gamma] \tag{3.4}$$

Hence it follows that along a time interval where the control is constant

$$\mathrm{sign}\,\dot{\varphi} = \pm\mathrm{sign}\,q$$

Note that the point $A^+$ has coordinates $(\varphi_0^+, 0) : -\Pi'(\varphi_0^+) + \gamma = 0$ (the point $A^-$ has coordinates $(\varphi_0^-, 0) : -\Pi'(\varphi_0^-) - \gamma = 0$), and is a singular point of the focus type for the perturbed system for fixed $u=\gamma(u=-\gamma)$.

We will now consider the optimal path for motion of the point $(\varphi_0, 0)$ to the boundary of the terminal set $\Omega$. Let $u=-\gamma$ in the first interval of constant control. Then the path beginning at $(\varphi_0, 0)$ lies in the lower half-plane: $\psi(t)<0$ for small $t$. Consequently, the control in the first interval corresponds to (2.1). Similarly one can show that the choice $u(0)=\gamma$ in the first interval leads to the same control law, and in both cases the path intersects the abscissa axis outside the segment $[A^-, A^+]$, due to the above property of the phase portrait. This ensures a positive numerator and denominator in (3.4), and consequently positive constant $C$ throughout the interval of motion.

In the proof, we have considered two optimal paths starting with $u=\pm\gamma$, which correspond to different extrema of the terminal functional, and either of them leads to escape from the potential well.

Hence when one considers the escape problem, it is sufficient to discuss only the method of control on the velocity gradient of the form (2.1). Consequently, to establish the possibility of escape, it is sufficient to check that it can occur using the VG method.

For this purpose we consider the Poincaré mapping for the flow $\varphi_t$: $\mathbb{R}^2 \to \mathbb{R}^2$, of the differential equation

$$\ddot{\varphi} + \rho\eta + \Pi'(\varphi) = \gamma\,\mathrm{sign}\,\dot{\varphi}$$

The basis (section) of the mapping is taken as the segment

$$\Sigma = \{(\varphi, \psi) \colon \varphi_0 \le \varphi \le \varphi_1, \psi = 0\}$$

where $\varphi_0$ is a point of stable equilibrium in the unperturbed system (the bottom of the potential well), while $\varphi_1$ is the point of intersection of the branches of the separatrix (Fig. 3). Then the mapping $P$: $\sum \to \sum$ itself is written as

$$P(\bar{\varphi}) = \pi\phi_T(\bar{\varphi}, 0), \quad \bar{\varphi} \in \Sigma \tag{3.5}$$

where $\pi$ is the projection on to the first argument and $T$ is the recurrence time to segment $\sum$.

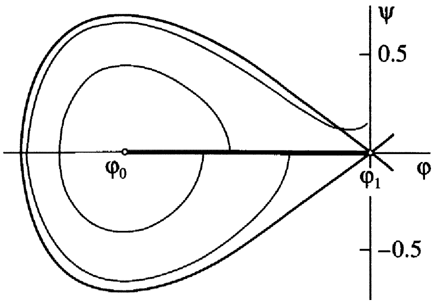


Fig. 3.

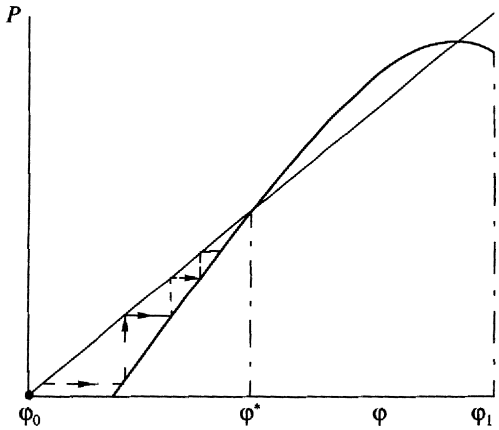


Fig. 4.

We will now show that if there is a fixed point of the Poincaré mapping (3.5) in the segment $[\varphi_0, \varphi_1]$, then it is impossible to transfer the system from the initial point $(\varphi_0, 0)$ to the final one $(\varphi_1, \dot{\varphi}_1)(\dot{\varphi}_1 > 0)$.

To prove this, we assume that in the segment $[\varphi_0, \varphi_1]$ there is at least one fixed point $P(\varphi^*) = \varphi^*$. Fig. 4 illustrates this situation. The motion of a point corresponding to the discrete Poincaré mapping with initial position $\varphi(0) = \varphi_0$ is shown by the dashed line. It can be seen that in this case it is impossible to transfer from the initial point to the final one $(\varphi_1, \dot{\varphi}_1)$, and the motion terminates when $\varphi = \varphi^*$.

Hence, we have proved the following theorem on controlled escape.□

**Theorem 2.** *Suppose the path of system* (1.1), *corresponding to velocity-gradient control* (2.1), *takes the system from the initial state* $(\varphi_0, 0)$ *to the final state* $(\varphi_1, \dot{\varphi}_1)(\dot{\varphi}_1 > 0)$ *and thus provides escape from the well in the sense of* Theorem 1, *i.e., the path intersects the separatrix. Then the segment* $[\varphi_0, \varphi_1]$ *does not contain fixed points of the Poincaré mapping, whose section will be the segment* $\sum$.

We have thus shown that if the use of control method (2.1) does not guarantee escape, then escape will not occur using any other control algorithm. Theorem 2 gives the necessary conditions for escape to occur.

## 4. Analysis of the problem for a Duffing oscillator

Let the potential function be

$$\Pi(\varphi) = -\varphi^2/2 + \varphi^4/4$$

To fix our ideas we will consider the left-hand equilibrium position $\varphi_0 = -1$. As the set $\Omega$ we will consider the right half-plane $\varphi \geq 0$. It is obvious that bringing the point to its boundary guarantees escape from the potential well.

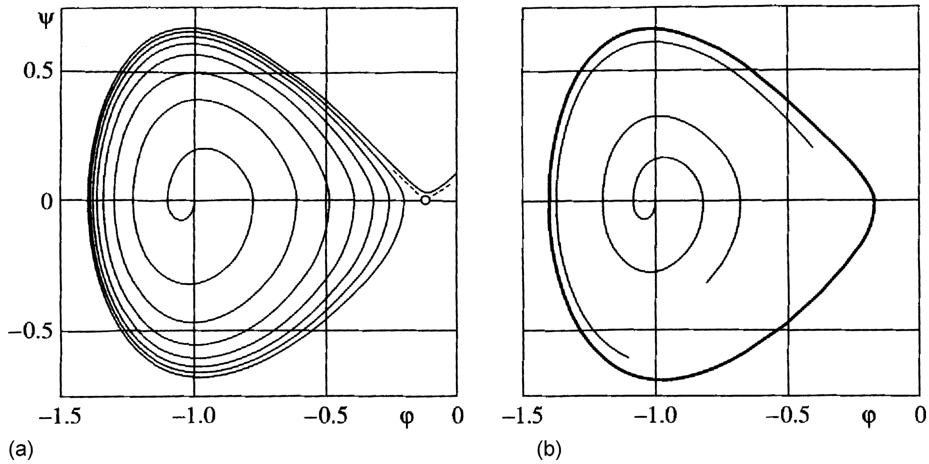


Fig. 5.

A numerical check shows that if $\rho = 0.25$ and $\gamma = 0.125$, escape is possible; the path for such escape is shown in Fig. 5a. We now fix the value of $\rho$ and reduce $\gamma$. Then for some (critical) value $\gamma^*$ a limit cycle occurs and the set $\Omega$ becomes unattainable, see Fig. 5b. This corresponds to the occurrence of a fixed point in the Poincaré mapping in the segment $[-1, 0]$.

Hence we can define a critical curve $\rho = f(\gamma)$ in the space of the parameters $(\rho, \gamma)$. For a given coefficient of friction $\rho$, escape is possible for any $\gamma > \gamma^*$ and is impossible for $\gamma \leq \gamma^*$, $\gamma^* = f^{-1}(\rho)$. Here $f^{-1}(\rho)$ is the function inverse to $f(\gamma)$. For this case of a Duffing potential, the critical curve can only be constructed by numerical methods, see Fig. 6. However, we can give an analytic construction of the asymptotic form of this curve in the vicinity of the origin, for which we write the energy balance equation for system (1.1) in the case of VG control. Multiplying this equation by $\dot{\varphi}$ and then integrating, get

$$V(t) = V(0) + \int_0^t (\rho\dot{\varphi}^2 - \gamma|\dot{\varphi}|)dt \tag{4.1}$$

where $V(t) = \dot{\varphi}^2/2 + \Pi(\varphi)$ is the energy of the system.

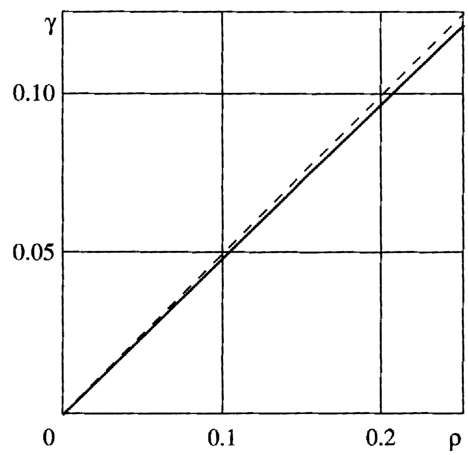


Fig. 6.

The critical situation corresponds to a limit cycle with period $T$. Then $V(T)=V(0)$, and Eq. (4.1) is rewritten as

$$\int_0^T (\rho\dot{\varphi}^2 - \gamma|\dot{\varphi}|)dt = 0 \tag{4.2}$$

If the perturbations in ρ and γ are sufficiently small, the motion of system (1.1) can be represented in the form

$$\varphi(t) = \varphi_h(t) + \rho\varphi_1(t) + \gamma\varphi_2(t) + o(\rho^2+\gamma^2)$$

where $(\varphi_h(t), \dot{\varphi}_h(t))$ is the trajectory of motion of a conservative system $(\rho=\gamma=0)$ with energy $h$. Taking only terms of first order of smallness in ρ and γ in (4.2) into account we get

$$\int_0^T (\rho\dot{\varphi}_h^2 - \gamma|\dot{\varphi}_h|)dt = 0$$

Then we obtain the function

$$\alpha(h) = \frac{\gamma}{\rho} = \left(\int_0^T |\dot{\varphi}_h| dt\right)^{-1} \int_0^T \dot{\varphi}_h^2 dt \tag{4.3}$$

showing for what ratio of ρ and γ it is possible for the system to be retained at a given energy level $h$ using arbitrarily small perturbations, introduced into the system.

It has been pointed out[16] that this approach was developed by Kirchhoff in 1850 and applied to problems of elastic stability by Bryan (1891) and Timoshenko (1906). It has been used[17] in the form of a "harmonic energy balance" subject to the condition that the periodic mode of the system is approximately harmonic.

We will apply to relation (4.3) the exact equations of periodic motion in a Duffing system. They take the form[9]

$$\varphi_k(t) = \frac{\sqrt{2}\,\mathrm{dn}(t_k, t)}{\sqrt{2-k^2}}, \quad \dot{\varphi}_k = -\frac{\sqrt{2}k^2\mathrm{sn}(t_k, k)\mathrm{cn}(t_k, k)}{\sqrt{2-k^2}}; \quad t_k = \frac{t}{\sqrt{2-k^2}} \tag{4.4}$$

and contain Jacobi elliptic functions[18,19], whose amplitude $k \in (0, 1)$ is related to the energy of the system by

$$h(k) = -\frac{1-k^2}{(2-k^2)^2}$$

The case $k=0$ corresponds to the equilibrium position of the system at the bottom of the well, while $k=1$ corresponds to motion in a homoclinic orbit, which is the boundary of the basin of attraction of the equilibrium point $(-1, 0)$.

Elliptic function theory shows that the period is given by

$$T(k) = 2\mathbf{K}(k)\sqrt{2-k^2}$$

where $\mathbf{K}(k)$ is the complete elliptic integral of the first kind.

Substituting (4.4) into (4.3) we get

$$\alpha(k) = \frac{k^4}{\sqrt{2}(2-k^2)^{3/2}(1-\sqrt{1-k^2})} \int_0^{T(k)} \mathrm{sn}^2(t_k, k)\mathrm{cn}^2(t_k, k)dt$$

The graph of $\alpha(h)$ (Fig. 7) shows where it is possible to maintain the system energy at the same level $h$ for the least ratio of the control margin $\gamma_h$ and coefficient of friction ρ. Consequently, transition to a higher energy level is possible for $\gamma > \gamma_h$. This implies that the critical $(\rho, \gamma^*)$ situation discussed above corresponds to the maximum point on the graph: $\alpha^* = \gamma^*/\rho = \max\alpha(h)$ (Fig. 7). An approximate value of $\alpha^*$ is 0.494 for $h^* = -0.0119$ (or $k^* = 0.988$). Then the tangent to the critical curve near zero is defined by $\gamma^* = \alpha^*\rho$ and is shown in Fig. 7 by the dashed line. The solid line shows the result from an exact numerical calculation performed in the MatLab system.

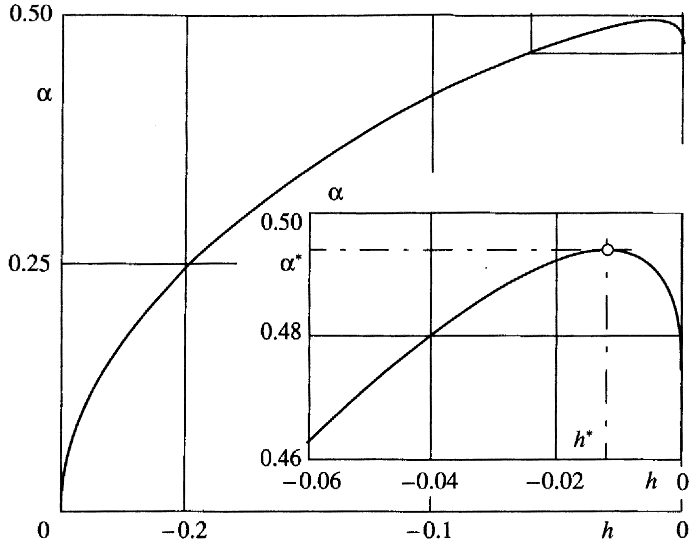


Fig. 7.

Similar arguments apply for a Helmholtz potential

$$\Pi(\varphi) = -\varphi^2/2 + \varphi^3/3$$

Here by escape from the well we mean the removal of a point from the basin of attraction of the stable equilibrium position $\varphi(0) = 1$, $\dot{\varphi}(0) = 0$. The critical curve is similar to that for a Duffing potential, but the slope of the tangent at zero is less and is defined by $\alpha^* = 0.413$, obtained numerically.


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